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On the Asymmetry of a Random Walk in the Presence of a Field

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Any ensemble of random walks with symmetric transition probabilities will have symmetric properties. However, any single realization of such a random walk may be asymmetric. In an earlier paper, Weiss and Weissman developed a measure of asymmetry and applied it to random walks in the absence of a field, showing that the degree of asymmetry (in the diffusion limit) is independent of time and that the most probable degree of asymmetry corresponds to the maximum possible. We show in the present paper how the presence of a symmetric field can change this result, both in making the degree of asymmetry depend on time, and driving the random walk toward a more symmetric state.

KEY WORDS: Random walks; extreme value theory; potential field.

1. INTRODUCTION

The probability distribution of the end-to-end distance of a random walk with symmetric transition probabilities will reflect that symmetry. However, it is well known that any single realization of a random walk can exhibit a considerable degree of asymmetry in spite of built-in symmetry properties. This was first remarked on by Kuhn[1] in the context of the theory of polymer configurations, but the basic idea is foreshadowed by Polya's criteria for the transience or recurrence of random walks to the origin.[2,3] Polya showed, for example, that a symmetric lattice random walk in one and two dimensions is certain to return to the origin, but the average return time for such a return is infinite. If we concentrate for the moment on the case of one-dimensional random walks, we see that this

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implies that any single realization of such a random walk which is initially at the origin tends to remain on one side of the origin for considerable periods of time before returning to the origin, thus leading to the appearance of asymmetry. Another indication of the asymmetry of individual random walks is contained in the arc-sine law,[4] versions of which were first discussed by Levý[5] and Sparre Andersen,[6] which also can be regarded as a quantitative measure of the tendency of a random walk to remain preferentially on one side of the origin. A number of investigators, motivated by problems related to configurational properties of polymer chains, have, within the past few years, analyzed various measures of asymmetry as applied to three-dimensional random walks.[7-12]

Recently Weiss and Weissman[13] have examined a simple measure of asymmetry in terms of maximum excursions of random walks. A random walk in one dimension at time t will have a maximum displacement in the negative x direction which we denote by $a(t)$ and a maximum positive displacement equal to $b(t)$. If one considers the probability density $g(\rho(t))$ of the random variable

$$\rho(t)=\frac{\min[a(t), b(t)]}{\max[a(t), b(t)]} \tag{1}$$

then it has been shown that (1) in the limit in which the random walk is replaced by a diffusion process, $g(\rho(t))$ is independent of time, (2) the maximum value of $g(\rho)$ occurs at $\rho=0$, which means that the most probable degree of asymmetry is the maximum possible and (3) the minimum value of $g(\rho)$ occurs at $\rho=1$. It should be noted that complete symmetry is equivalent to the result $g(\rho)=\delta(\rho-1)$. Generalizations of these results were also developed for random walks in higher dimensions.

In the present note we consider a generalization of the problem set in the last paragraph, in which the ordinary random walk in one dimension is replaced by a diffusion process in the presence of a biassing field symmetric around the origin, which attracts the particle to the origin. It will be assumed that the diffusion process takes place on the entire line. Let the time between successive returns to the origin be termed a sojourn and let the (random) maximum displacement during this time be denoted by L. We examine a class of bias fields which have the property that when $\langle L\rangle=\infty$ the function $g(\rho(t))$ has its maximum value at $\rho=0$, while when $\langle L\rangle$ is finite it attains a maximum value at a value of ρ intermediate between 0 and 1. Our final example is that of a diffusion process in a piecewise constant symmetric field with a constant diffusion coefficient. We show that for this class of random walks the asymptotically most probable value of ρ is $\rho=1$, i.e., the random walk is asymptotically completely symmetric.

2. ANALYSIS

In order to calculate the pdf of $\rho(t)$, we need first to calculate the joint pdf of the variables $a(t)$ and $b(t)$. Because of our assumption that the mean time to reach the origin from any point on the line is finite, we can, for example, let L_j be the maximum displacement in the positive x direction during the jth sojourn on the positive side of the origin. The limit $t \to \infty$ can be identified with the limit $n(t) \to \infty$, where $n(t)$ is the number of sojourns on the positive half-line in time t. With this notation we can, as an approximation, represent $b(t)$ in the form

$$b(t) \sim \max(L_1, L_2, \ldots, L_{n(t)}) = b_n \tag{2}$$

with an error that tends to 0 in the limit $t \to \infty$. The difference between the actual $b(t)$ and the approximation given in this last equation is due to diffusion processes in which there is a sojourn on the positive half-line that has not been completed by time t. Rather than develop an expression for the time-dependent pdf of $\rho(t)$, we will calculate an approximation for the pdf of ρ_n, defined, in analogous fashion to Eq. (1), to be

$$\rho_n = \frac{\min(a_n, b_n)}{\max(a_n, b_n)} \tag{3}$$

that is to say, our measure of time will be the number of sojourns, rather than the time.

In the limit $t \to \infty$ the number of sojourns n on either side of the origin will, with a probability approaching 1, also approach ∞. Furthermore, the random variable a_n will be independent of b_n. Hence, if the pdf of a_n is denoted by $f_n(a)$, then the joint distribution of a_n and b_n, $p_n(a, b)$, can be expressed as

$$p_n(a, b) = f_n(a) f_n(b) \tag{4}$$

with the result that the pdf of ρ_n is[13]

$$g_n(\rho) = 2 \int_0^\infty a f_n(a) f_n(\rho a)\, da \tag{5}$$

The factor of 2 arises from the two possibilities that either a or b can be the minimum. Thus, our problem reduces to that of calculating $f_n(a)$. Let us suppose that we know the pdf of the maximum displacement during a single sojourn on either the positive or negative x axis (the two pdf's are necessarily the same, since we assume the random walk to be symmetric). This function will be denoted by $p(L)$. Then, since $n \gg 1$, we may use asymptotic results from extreme-value theory[14] to find an approximation to $f_n(a)$.

The result required for our further analysis can be stated in terms of a probability $H(l)$, defined by

$$H(l) = \Pr\{L \leqslant l\} \tag{6}$$

The average maximum displacement is expressed in terms of this probability as

$$\langle L \rangle = \int_0^\infty [1 - H(l)]\, dl \tag{7}$$

We assume that the bias is such as to ensure that $\langle L \rangle$ is finite. In free diffusion it is readily shown that $\langle L \rangle = \infty$. We will show how to calculate this parameter in the presence of a biassing field. Before doing so, we must address the question of how to set the initial conditions in defining what we mean by a sojourn. A sojourn on one side of the x axis, say the positive side, begins when a diffusing particle crosses the axis from the negative x axis. Thus, in a diffusion picture it is necessary to place the particle at the origin initially. The sojourn is terminated when the particle reaches the origin a second time. A trapping boundary condition can be imposed at $x = 0$ to terminate the sojourn, but if $x = 0$ is regarded as a trap initially, the particle will never move away from the origin. Further, a particle reaching the origin need not cross it to the line of opposite parity. The simplest way to take these considerations into account is to make the origin a partially reflecting point. The specific value of the reflection coefficient will have no bearing on our qualitative results, and therefore need not be assigned (although this could be done by passing to the continuum limit from a lattice random walk). The probability density for the position of the diffusing particle will be denoted by $p(x, t)$, which is the solution to

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left(D(x)\frac{\partial p}{\partial x}\right) - \frac{\partial}{\partial x}(v(x)\, p) = \mathscr{L}p \tag{8}$$

which is the most general evolution equation for a diffusion process allowing for the possibility of a spatially varying diffusion and field.

As a first step we calculate the probability that the maximum displacement in the course of a single sojourn is $\leqslant l$. This probability can be expressed in terms of the splitting probability $\varepsilon(l)$, which is defined to be the probability that the particle returns to the origin and is absorbed there without having reached $L = l$. This second condition can be assured by making $L = l$ a trapping point. It is known[15] that $\varepsilon(l)$ satisfies the equation adjoint to the operator $\mathscr{L}$ appearing on the right-hand side of Eq. (8):

$$\mathscr{L}^{\dagger}\varepsilon = D(x)\frac{d^2\varepsilon}{dx^2} + v(x)\frac{d\varepsilon}{dx} = 0 \tag{9}$$

which is to be solved subject to the boundary conditions[16]

$$\left.\frac{d\varepsilon}{dx}\right|_{x=0} = \kappa[\varepsilon(0) - 1], \qquad \varepsilon(l) = 0 \tag{10}$$

The first of these corresponds to $x = 0$ being a partially reflecting point, the parameter κ indicating the degree of absorption. When $\kappa = \infty$, $x = 0$ is a perfect trap. The function $H(l)$ is expressed in terms of $\varepsilon(l)$ as

$$H(l) = \varepsilon(0) \tag{11}$$

Equation (9) is readily solved with the boundary conditions in Eq. (10) to yield the result

$$H(l) = 1 - \frac{1}{1 + \kappa\Omega(l)} \tag{12}$$

where

$$\Omega(l) = \int_0^l \exp\left[-\int_0^x \frac{v(z)}{D(z)}\,dz\right]dx \tag{13}$$

In particular, we may write the expression for $\langle L\rangle$ in terms of $\Omega(l)$ as

$$\langle L\rangle = \int_0^\infty \frac{dl}{1 + \kappa\Omega(l)} \tag{14}$$

from which it is evident that the question of the convergence of the integral can be settled without knowing the parameter κ. It follows from the combination of Eqs. (13) and (14) that when $v(x) \geqslant 0$, $\langle L\rangle = \infty$, as is otherwise obvious.

Let us examine the consequences of the assumption that for large l, the function $\Omega(l)$ goes like

$$\Omega(l) \sim (l/l_0)^\alpha \tag{15}$$

where l_0 is a constant. This class of $\Omega(l)$ has the property that $\langle L\rangle = \infty$ when $\alpha \leqslant 1$ and $\langle L\rangle$ is finite for $\alpha > 1$. In order to find the probability that the largest of n maximum displacements, denoted by L_m, is $\leqslant l$, we use the following result from extreme-value theory[14]: Assume that for all $x > 0$ we have

$$\lim_{t \to \infty} \frac{1 - H(tx)}{1 - H(t)} = x^{-\gamma} \tag{16}$$

where γ is a positive constant. Then there is a sequence $b_n > 0$ such that, as $n \to \infty$,

$$\Pr\{L_m \leqslant b_n x\} = \exp\left(-\frac{1}{x^{\gamma}}\right) \tag{17}$$

The constants $\{b_n\}$ can be chosen as the solution to

$$H(b_n) = 1 - \frac{1}{n} \tag{18}$$

We see from the combination of Eqs. (12) and (15) that in the present instance the parameters γ and b_n are given by

$$\gamma = \alpha, \qquad b_n = \frac{l_0}{\kappa^{1/\alpha}} n^{1/\alpha} \tag{19}$$

The probability density function $f_n(a)$ that appears in Eq. (4) is therefore given by

$$f_n(a) = \frac{\alpha b_n^{\alpha}}{a^{\alpha+1}} \exp\left[-\left(\frac{b_n}{a}\right)^{\alpha}\right] \tag{20}$$

Upon inserting this expression into Eq. (4) and evaluating the resulting integral, we find that the function $g_n(\rho)$ is

$$g_n(\rho) = \frac{2\alpha\rho^{\alpha-1}}{(\rho^{\alpha}+1)^2}, \qquad \rho \leqslant 1 \tag{21}$$

We see that just as in the case of field-free diffusion, $g_n(\rho) = g(\rho)$ is independent of n. When $\alpha \leqslant 1$, which corresponds to $\langle L \rangle = \infty$, the location of the maximum of $g(\rho)$ occurs at $\rho = 0$, while for $\alpha > 1$ or $\langle L \rangle < \infty$, the most likely value of ρ occurs at

$$\rho_m = \left(\frac{\alpha - 1}{\alpha + 1}\right)^{1/\alpha} \tag{22}$$

which approaches 1 as $\alpha \to \infty$, but is strictly less than 1 when α is finite. Notice that the particular case in which $\Omega(l)$ has the asymptotic ($l \to \infty$) power law behavior shown in Eq. (15) it follows that when $\langle L \rangle$ is finite

$$v(x) \sim -\left(\frac{\alpha-1}{x}\right) D(x) \tag{23}$$

for x sufficiently large.

It is interesting to consider an asymptotically constant bias field, i.e., one that has the property

$$\frac{v(x)}{D(x)} \sim -\frac{1}{l_0}, \qquad x \to \infty \tag{24}$$

This relation implies that $v(x)$ and $D(x)$ have the same functional dependence in the limit $x \to \infty$. When Eq. (24) is valid, $\Omega(l) \sim l_0 \exp(l/l_0)$ and $\langle L \rangle$ is finite. A different extreme-value distribution is required for the calculation of the asymptotic form of $f_n(a)$. This is the following[14]: Under the assumption that $\langle L \rangle < \infty$ one defines the function

$$R(t) = \frac{\int_t^\infty [1 - H(l)]\, dl}{1 - H(t)} \tag{25}$$

If it is the case that

$$\lim_{t \to \infty} \frac{1 - H(t + xR(t))}{1 - H(t)} = e^{-x} \tag{26}$$

then there exist constants a_n and b_n such that

$$\lim_{n \to \infty} \Pr\{L_m < a_n + b_n x\} = \exp[-e^{-x}] \tag{27}$$

The constant a_n is given by the solution to Eq. (18) and b_n can be set equal to $R(a_n)$. It is readily verified that $R(t) \sim l_0$ and that the condition given in Eq. (26) is satisfied. Hence the probability density for the largest of n displacements is given by

$$f_n(a) \sim \frac{n}{\kappa l_0^2} \exp\left[-\frac{a}{l_0} - \frac{n}{\kappa l_0} \exp\left(-\frac{a}{l_0}\right)\right] \tag{28}$$

with the result that

$$g_n(\rho) \sim \frac{2n^2}{(\kappa l_0)^2} \int_{-\infty}^{\infty} v \exp\left[-v(1+\rho) - \frac{n}{\kappa l_0}(e^{-v} + e^{-v\rho})\right] dv \tag{29}$$

$$= \frac{2n^2}{(\kappa l_0)^2} \int_0^\infty \beta^\rho \ln\left(\frac{1}{\beta}\right) \exp\left[-\frac{n}{\kappa l_0}(\beta + \beta^\rho)\right] d\beta \tag{30}$$

One can find an asymptotic formula for $g_n(\rho)$ in the limit in which $n \gg \kappa l_0$. For convenience we set $\lambda = n/(\kappa l_0)$, so that the asymptotic representation will be valid when $\lambda \gg 1$. Let us set

$$\beta + \beta^\rho = \xi \tag{31}$$

with an inverse function denoted by $\beta = f(\xi)$. These substitutions allow us to express $g_n(\rho)$ as the Laplace transform

$$g_n(\rho) = 2\lambda^2 \int_0^\infty [f(\xi)]^\rho f'(\xi) \ln\left(\frac{1}{f(\xi)}\right) e^{-\lambda\varepsilon} \, d\xi \tag{32}$$

Since we are interested in large values of λ, we can use an Abelian theorem for Laplace transforms[17] to assert that the major contribution to the value of the integral comes from the behavior of the integrand at small ξ. In that region we may solve Eq. (31) to find that to lowest order

$$f(\xi) \sim \xi^{1/\rho} \tag{33}$$

When this is substituted into Eq. (32), one finds

$$\begin{aligned} g_n(\rho) &\sim \frac{2\lambda^2}{\rho^2} \int_0^\infty \xi^{1/\rho} \ln\left(\frac{1}{\xi}\right) e^{-\lambda\xi} \, d\xi \\ &= \frac{2\lambda^{1-1/\rho}}{\rho^2} \Gamma\left(\frac{1}{\rho}+1\right)\left[\psi\left(\frac{1}{\rho}\right) - \ln\lambda\right] \end{aligned} \tag{34}$$

in which $\psi(x) = d \ln \Gamma(x)/dx$. It is possible to find the properties of $g_n(\rho)$ at the endpoints $\rho = 0$ and $\rho = 1$. When $\rho = 0$ we have, after some reduction,

$$g_n(0) \sim \frac{2n}{\kappa l_0} e^{-n/\kappa l_0} \int_0^\infty \left[\ln\left(\frac{n}{\kappa l_0}\right) - \ln u\right] e^{-u} \, du \sim \ln\left(\frac{n}{\kappa l_0}\right) \frac{2n}{\kappa l_0} e^{-n/\kappa l_0} \tag{35}$$

which goes to 0 with increasing n. On the other hand, the approximate representation for $g_n(1)$ is, from Eq. (30), given by

$$\begin{aligned} g_n(1) &\sim \frac{2n^2}{\kappa^2 l_0^2} \int_{-\infty}^\infty v \exp\left(-2v - \frac{2n}{\kappa l_0} e^{-v}\right) dv \\ &= \frac{1}{2} \int_0^\infty \xi e^{-\xi} \ln\left(\frac{2n}{\kappa l_0 \xi}\right) d\xi \sim \frac{\ln(2n)}{2} \end{aligned} \tag{36}$$

which increases logarithmically as n increases. A plot of $g_n(\rho)$ as a function of ρ is shown in Fig. 1, from which one can observe that for any finite n

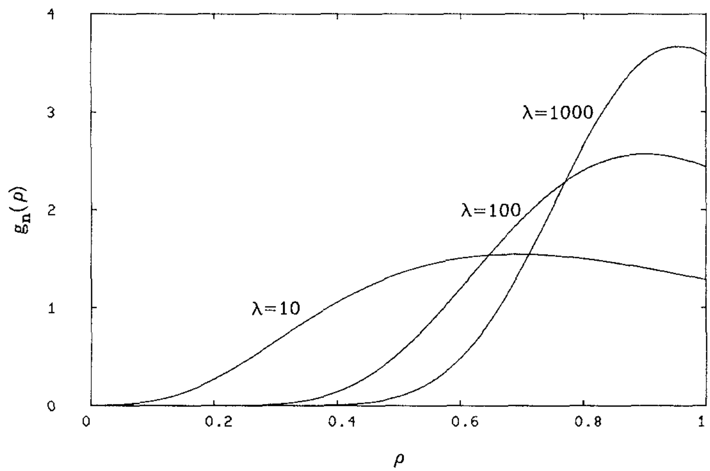

Fig. 1. Three graphs of the function $g_n(\rho)$ defined in Eq. (30) plotted as a function of the parameter $\lambda = n/(\kappa l_0)$. Notice the slow approach of the maximum to $\rho = 1$ as a function of λ.

the peak occurs at a value of $\rho < 1$, but as $n \to \infty$ the position of the peak approaches $\rho = 1$. Qualitatively similar results can be found when the functions $v(x)/D(x)$ have the more general asymptotic behavior

$$\frac{v(x)}{D(x)} \sim \left(\frac{x}{l_0}\right)^m, \qquad x \to \infty \tag{37}$$

for any $m > 0$.

We therefore see that the presence of a biassing field, or in physical terms, a potential, can lead to qualitative changes in the asymmetry, as measured by the random variable ρ_n. In the case in which the diffusion coefficient $D(x)$ is a constant and $v(x)$ is constant on either side of the origin with $v(x) = -v(-x)$, the diffusion process or random walk will be asymptotically symmetric. We have shown that the parameter whose properties are easiest to calculate in the context of asymmetry is the maximum displacement in a given direction, and that the rate at which the cumulative probability $H(l)$ approaches 1 as $l \to \infty$ determines the qualitative properties of the asymmetry. It would be desirable to be able to relate the asymmetry to a return time to the origin, but a complete solution along those lines is more complicated than the one discussed in the present note. Our analysis is one dimensional; the extension to higher dimensions poses a much more difficult mathematical problem. To our knowledge there have been no earlier studies of the problem of the asymmetry of symmetric random walks which include the effects of a field.

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